

CRITICAL FIRST-PASSAGE PERCOLATION STARTING ON THE BOUNDARY

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ABSTRACT. We consider first-passage percolation on the two-dimensional triangular lattice \mathcal{T} . Each site $v \in \mathcal{T}$ is assigned independently a passage time of either 0 or 1 with probability $1/2$. Denote by $B^+(0, n)$ the upper half-disk with radius n centered at 0, and by c_n^+ the first-passage time in $B^+(0, n)$ from 0 to the half-circular boundary of $B^+(0, n)$. We prove

$$\lim_{n \rightarrow \infty} \frac{c_n^+}{\log n} = \frac{\sqrt{3}}{2\pi} \text{ a.s.}, \quad \lim_{n \rightarrow \infty} \frac{Ec_n^+}{\log n} = \frac{\sqrt{3}}{2\pi}, \quad \lim_{n \rightarrow \infty} \frac{\text{Var}(c_n^+)}{\log n} = \frac{2\sqrt{3}}{\pi} - \frac{9}{\pi^2}.$$

Our proof first derives the distribution of “conformal radii” for nested half-loops (from the scaling limit of percolation interfaces) surrounding 0, which may be of independent interest. The results about c_n^+ enable us to prove limit theorems with explicit constants for any first-passage time between boundary points (with left and right tangent lines) of Jordan domains. In particular, we find the explicit limit theorems for the cylinder point to point and cylinder point to line first-passage times.

1. INTRODUCTION

Since Hammersley and Welsh [9] introduced first-passage percolation (FPP) in 1965, this stochastic growth model has attracted much attention from mathematicians and physicists. For the main results and recent developments of FPP, we refer the reader to the survey [1], especially Section 3.7.1 there for critical FPP. For Bernoulli critical FPP on the triangular lattice, the author in [22] derived the exact asymptotic behavior for the first-passage time from the center of a disk to its boundary. In this paper, we consider a boundary version of that result. Namely, we study the first-passage time in a half-disk from its center to its half-circular boundary.

Here is an alternative description of our model. Construct a random maze on the hexagonal lattice by putting an obstacle on each hexagon according to the outcome of a fair coin toss. Consider a walker in the maze starting at the hexagon centered at 0. What is the minimum number of obstacles the walker has to cross to reach the circle of radius n centered at 0 (interior point version)? Or, if the walker is only allowed to walk in the upper half-plane, what is the minimum number of obstacles the walker has to cross to reach the upper half-circle of radius n centered at 0 (boundary point version)? From the results in [22] and this paper, one can see that when n is large with high probability the latter quantity is approximately 3 times the former. See Figure 1 for an illustration.

Let $\mathcal{T} = (\mathbf{V}, \mathbf{E})$ be the two-dimensional triangular lattice, where $\mathbf{V} = \{x + ye^{i\pi/3} : x, y \in \mathbb{Z}\}$ is the set of sites, and $\mathbf{E} = \{\{x, y\} : x, y \in \mathbf{V}, \|x - y\|_2 = 1\}$ is the set of bonds. Suppose $\{t(v) : v \in \mathbf{V}\}$ is a family of i.i.d. Bernoulli random variables: each $t(v)$ takes the value 0 or 1 with equal probability. This is **critical site percolation** on \mathcal{T} , but we also call it **critical first-passage percolation** on \mathcal{T} for reasons that will be clear below. We write P for this critical percolation measure, and E for the corresponding expectation. A **path** is a sequence (v_0, v_1, \dots, v_n) of distinct sites of \mathcal{T} such that $\{v_{j-1}, v_j\} \in \mathbf{E}$ for

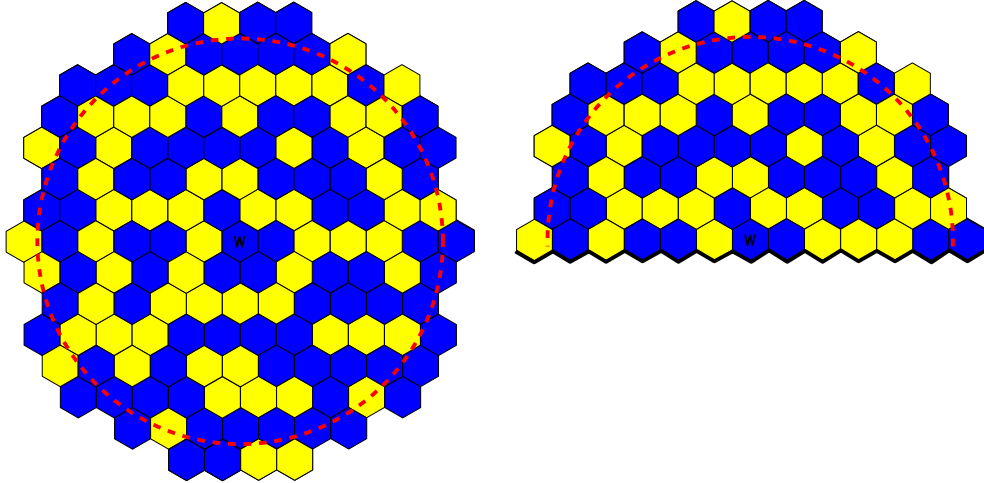


FIGURE 1. All blue hexagons are void spaces and all yellow hexagons are obstacles. W means that the walker starts at the hexagon centered at 0. The walker has to cross at least 1 obstacle to reach the red circle in the left maze, and cross at least 2 obstacles to reach the red half-circle in the right maze.

each $j = 1, 2, \dots, n$. For a path $\gamma = (v_0, v_1, \dots, v_n)$, define the passage time of γ as

$$T(\gamma) = \sum_{j=0}^n t(v_j).$$

We also consider the dual of \mathcal{T} , the two-dimensional hexagonal lattice $\mathcal{H} = (\mathbf{V}_d, \mathbf{E}_d)$, such that each $v \in \mathbf{V}$ lies at the center of exactly one face (or hexagon) of \mathcal{H} . Each hexagon is assigned the same value as its center (which is a site in \mathbf{V}). For $n \in \mathbb{N}$, let $B^+(0, n)$ be the smallest connected domain of hexagons containing the closed upper half-disk centered at 0 with radius n . Let $\Delta_o B^+(0, n)$ be the half-circular boundary of $B^+(0, n)$, that is, the set of hexagons that do not belong to $B^+(0, n)$ but are adjacent to those hexagons (with centers lie on or above x -axis) intersecting the half-circle of radius n centered at 0. Denote by c_n^+ the first-passage time in $B^+(0, n)$ between 0 and $\Delta_o B^+(0, n)$, more precisely,

$$c_n^+ := \inf\{T(\gamma) : \gamma \in B^+(0, n) \text{ starting at 0 and ending at a neighbor of } \Delta_o B(0, n)\}.$$

Our main theorem is

Theorem 1.

$$\lim_{n \rightarrow \infty} \frac{c_n^+}{\log n} = \frac{\sqrt{3}}{2\pi} \text{ a.s.}, \quad \lim_{n \rightarrow \infty} \frac{Ec_n^+}{\log n} = \frac{\sqrt{3}}{2\pi}, \quad \lim_{n \rightarrow \infty} \frac{\text{Var}(c_n^+)}{\log n} = \frac{2\sqrt{3}}{\pi} - \frac{9}{\pi^2}.$$

Remark 1. For FPP corresponding to other critical percolation models (e.g., bond percolation on the square lattice [8] and Voronoi percolation [3]), it is expected that Theorem 1 still holds, provided that the convergence of chordal exploration path to chordal SLE_6 is established. The reason is the following: similarly to the proof of Proposition 3, one can show that c_n^+ equals the maximum number of disjoint yellow half-circuits surrounding 0 in $B^+(0, n)$; then using the idea from Section 4 of [2], one can express the event $\{c_n^+ \geq k\}$ in terms of the collection of all cluster interfaces.

Remark 2. It should be possible to use the argument in [12] (see also [7]) and Theorem 1 above to prove the following central limit theorem:

$$\frac{c_n^+ - \sqrt{3} \log n / (2\pi)}{\sqrt{(2\sqrt{3}/\pi - 9/(\pi^2)) \log n}} \rightarrow N(0, 1) \text{ in distribution as } n \rightarrow \infty,$$

where $N(0, 1)$ is a standard normal random variable (with mean 0 and variance 1).

Let $\mathcal{H}_\delta := \delta\mathcal{H}$ be the two-dimensional hexagonal lattice with lattice spacing δ . For $\alpha \in (0, 2\pi)$, let $\mathbb{D}^\alpha := \{re^{i\theta} : 0 < r < 1, 0 < \theta < \alpha\}$ be the circular sector with center angle α . Let \mathbb{D}_δ^α be the smallest connected domain of hexagons (with lattice spacing δ) containing $\overline{\mathbb{D}^\alpha}$. Consider critical percolation on \mathbb{D}_δ^α and write E_δ for the corresponding expectation. Let 0_δ be the closest hexagon of \mathbb{D}_δ^α to 0. Then $0_\delta \equiv 0$ by definition. Define $c_\delta^+(\alpha)$ to be the first-passage time in \mathbb{D}_δ^α between 0_δ and the part of circular boundary of \mathbb{D}_δ^α . Then Theorem 1 can be generalized to the following:

Corollary 1.

$$\lim_{\delta \downarrow 0} \frac{c_\delta^+(\alpha)}{-\log \delta} = \frac{\sqrt{3}}{2\alpha} \text{ a.s.}, \quad \lim_{\delta \downarrow 0} \frac{E_\delta c_\delta^+(\alpha)}{-\log \delta} = \frac{\sqrt{3}}{2\alpha}, \quad \lim_{\delta \downarrow 0} \frac{\text{Var}_\delta(c_\delta^+(\alpha))}{-\log \delta} = \frac{2\sqrt{3}}{\alpha} - \frac{9}{\pi\alpha}.$$

Suppose $D \subsetneq \mathbb{C}$ is a Jordan domain and $a, b \in \partial D$. Let $\{\beta(t) : t \geq 0\}$ be the arc-length parametrization of ∂D . We assume ∂D has both left and right tangent lines at a and b , i.e., $\beta(t)$ has left and right derivatives at t_0 and t_1 where $\beta(t_0) = a$ and $\beta(t_1) = b$. Let $\Theta_D(a)$ be the angle subtended by the left and right tangent lines at a . Then $\Theta_D(a) \in (0, 2\pi)$ and whether $\Theta_D(a) > \pi$ or not can be determined easily by comparing a small enough neighborhood of a in D with a circular sector. Define $\Theta_D(b)$ in the same way. Let D_δ be the smallest connected domain of hexagons containing \overline{D} . Denote by $T_{D_\delta}(a_\delta, b_\delta)$ the first-passage time in D_δ between a_δ and b_δ . Then we have

Proposition 1.

$$\lim_{\delta \downarrow 0} \frac{T_{D_\delta}(a_\delta, b_\delta)}{-\log \delta} = \frac{\sqrt{3}}{2\Theta_D(a)} + \frac{\sqrt{3}}{2\Theta_D(b)} \text{ in probability}, \quad \lim_{\delta \downarrow 0} \frac{E_\delta T_{D_\delta}(a_\delta, b_\delta)}{-\log \delta} = \frac{\sqrt{3}}{2\Theta_D(a)} + \frac{\sqrt{3}}{2\Theta_D(b)}.$$

For $k \in \mathbb{Z}$, let

$$H_k = \{v \in \mathbf{V} : \text{Im}(v) = \sqrt{3}k\}$$

be a hyperplane. As in [20] and [11] (see also [9]), for $m < n$ we define

$$t_{m,n} = \inf\{T(\gamma) : \gamma \text{ is a path from } (0, \sqrt{3}m) \text{ to } (0, \sqrt{3}n), \text{ except for its endpoints, lies strictly between } H_m \text{ and } H_n\},$$

$$s_{m,n} = \inf\{T(\gamma) : \gamma \text{ is a path from } (0, \sqrt{3}m) \text{ to some point in } H_n, \text{ except for its endpoints, lies strictly between } H_m \text{ and } H_n\}.$$

$t_{0,n}$ and $s_{0,n}$ are called the **cylinder point to point** and **cylinder point to line first-passage times**, respectively. Then we have

Proposition 2.

$$\lim_{n \rightarrow \infty} \frac{s_{0,n}}{\log n} = \frac{\sqrt{3}}{2\pi} \text{ a.s.}, \quad \lim_{n \rightarrow \infty} \frac{E s_{0,n}}{\log n} = \frac{\sqrt{3}}{2\pi}, \quad \lim_{n \rightarrow \infty} \frac{\text{Var}(s_{0,n})}{\log n} = \frac{2\sqrt{3}}{\pi} - \frac{9}{\pi^2}.$$

$$\lim_{n \rightarrow \infty} \frac{t_{0,n}}{\log n} = \frac{\sqrt{3}}{\pi} \text{ in probability but not a.s.},$$

$$\lim_{n \rightarrow \infty} \frac{Et_{0,n}}{\log n} = \frac{\sqrt{3}}{\pi}, \quad \lim_{n \rightarrow \infty} \frac{\text{Var}(t_{0,n})}{\log n} = \frac{4\sqrt{3}}{\pi} - \frac{18}{\pi^2}.$$

Our proof strategy for Theorem 1 is analogous to that of [22] where the first-passage time from the center of a disk to its boundary is studied. Using a color switching technique we obtain that the first-passage time in a half-annulus has the same distribution as the number of (cluster) interface half-loops surrounding 0 in the half-annulus under a monochromatic boundary condition. The well-known result that the percolation chordal exploration path converges weakly to chordal SLE₆ (see [19] and [5]) tells us that those discrete interface half-loops converges weakly to the corresponding half-loops in the continuum. Then we use SLE techniques from [14] to compute the distribution of the “conformal radii” of the nested half-loops surrounding a fixed point. Surprisingly, this distribution is related to the distribution of conformal radii of CLE_{24/5} whose moment generating function is derived in [17]. This allows us to obtain explicit limit theorems for the half-loops in the continuum. Then the limit results for c_n^+ and $E[c_n^+]$ follow easily. Using Hongler and Smirnov’s formula for the expected number of clusters in a rectangle [10], we give an alternative and more straightforward proof (using no SLE techniques) of the limit results for c_n^+ and $E[c_n^+]$. Let us mention that it is possible to use the SLE techniques in Section 4.3 of [6] to give a third proof. In order to prove the limit result for $\text{Var}(c_n^+)$, we use a martingale method from [12].

The organization of the paper is as follows. In Section 2, we give some definitions and relate the first-passage time in an annulus to the number of interface half-loops. In Section 3, we compute the exact distribution of “conformal radii” for the interface half-loops in the continuum. In Sections 4 and 5, we present two different proofs for the a.s. and L^1 convergences of $c_n^+/\log n$. In Section 6, we show the convergence of $\text{Var}(c_n^+)/\log n$. In the last section, we complete the proofs of Theorem 1, Corollary 1, Propositions 1 and 2.

2. PRELIMINARIES

2.1. Definitions and some discrete results. Recall that $\mathcal{T} = (\mathbf{V}, \mathbf{E})$ is the two-dimensional triangular lattice and $\mathcal{H} = (\mathbf{V}_d, \mathbf{E}_d)$ is its dual. The critical site percolation on \mathcal{T} is an assignment of 0 (equivalently, blue) or 1 (equivalently, yellow) to each site of \mathcal{T} (i.e., to each hexagon of \mathcal{H}). We denote the resulting probability space by (Ω, \mathcal{F}, P) where $\Omega = \{0, 1\}^{\mathbf{V}}$, and write E for the corresponding expectation. Two hexagons are neighbors if they share a common edge. So a path (v_0, v_1, \dots, v_n) in \mathcal{T} corresponds to a path (h_0, h_1, \dots, h_n) in \mathcal{H} such that each v_i lies at the center of the hexagon h_i for each $0 \leq i \leq n$. A path is called a **circuit** if its first and last sites (or hexagons) are neighbors.

We denote by \mathbb{D} the unit disk in \mathbb{C} centered at 0. Define \mathbb{H} to be the upper half-plane, i.e., $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. Let $\mathbb{D}^+ := \mathbb{D} \cap \mathbb{H}$ be the upper half unit disk. For $r > 0$, let $\mathbb{D}_r^+ := r\mathbb{D}^+$ be the upper half-disk of radius r centered at 0. For $v \in \mathbf{V}$, denote by $B^+(v, r)$ the smallest connected domain of hexagons (in \mathcal{H}) which contains $\overline{v + \mathbb{D}_r^+}$. For $1 \leq r < R$, let $A^+(r, R)$ be the discrete half-annulus centered at 0 with inner radius r and outer radius R . More precisely,

$$A^+(r, R) := B^+(0, R) \setminus B^+(0, r).$$

Let $\Delta A^+(r, R)$ be the **external site boundary** of $A^+(r, R)$, i.e., the set of hexagons that do not belong to $A^+(r, R)$ but are adjacent to hexagons in $A^+(r, R)$. $\Delta A^+(r, R)$ contains two paths of hexagons which lie under the x -axis (see Figure 2). We denote the left path by $\Delta_l A^+(r, R)$ and the right one by $\Delta_r A^+(r, R)$. The set of hexagons that are in $\Delta A^+(r, R) \setminus \{\Delta_l A^+(r, R) \cup \Delta_r A^+(r, R)\}$ and intersect \mathbb{D}_r^+ is denoted by $\Delta_i A^+(r, R)$, and $\Delta A^+(r, R) \setminus \{\Delta_l A^+(r, R) \cup \Delta_r A^+(r, R) \cup \Delta_i A^+(r, R)\}$ is denoted by $\Delta_o A^+(r, R)$.

A path (v_0, v_1, \dots, v_n) in $A^+(r, R)$ is called a **half-circuit surrounding 0** if v_0 has a neighbor in $\Delta_l A^+(r, R)$ and v_n has a neighbor in $\Delta_r A^+(r, R)$. A **percolation cluster** is a maximal, connected and monochromatic subset of \mathcal{T} (or \mathcal{H}). An **interface path** is a sequence (e_0, e_1, \dots, e_n) of distinct edges of \mathcal{H} which belongs to the boundary of a cluster and e_{i-1} and e_i sharing a vertex of \mathcal{H} for each $i = 0, 1, \dots, n$. An **interface half-loop surrounding 0** is defined in the obvious way. Define

$\rho^+(r, R) :=$ the maximum number of disjoint yellow half-circuits surrounding 0
in $A^+(r, R)$,

$N^+(r, R) :=$ the number of interface half-loops surrounding 0 in $A^+(r, R)$,

$T^+(r, R) := \inf\{T(\gamma) : \gamma \in A^+(r, R), \text{ the first site of } \gamma \text{ has a neighbor in } \Delta_i A^+(r, R) \\ \text{and the last site of } \gamma \text{ has a neighbor in } \Delta_o A^+(r, R)\}.$

As Proposition 2.4 in [22], we have

Proposition 3. *Suppose $1 \leq r < R$. Then we have*

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- $T^+(r, R) = \rho^+(r, R).$
- *Assume hexagons in $\Delta_o A^+(r, R)$ are blue. Then $T^+(r, R)$ has the same distribution as $N^+(r, R)$.*

Proof. The proof is similar to that of Proposition 2.4 in [22]. Since the proof of the first item is standard (see, e.g., (2.39) in [12]), we describe the main idea for the proof of the second item. The key idea is to find a bijection between $\{\rho^+(r, R) = n\}$ and $\{N^+(r, R) = n\}$ for any $n \in \mathbb{N} \cup \{0\}$. The case when $n = 0$ is trivial since the bijection is just the identity map. So we may just assume $n \geq 1$. Let $\omega \in \{\rho^+(r, R) = n\}$. We label all disjoint yellow half-circuits surrounding 0 in $A^+(r, R)$ from outside to inside by $\mathcal{C}_1(\omega), \mathcal{C}_2(\omega), \dots, \mathcal{C}_n(\omega)$. More precisely, $\mathcal{C}_1(\omega)$ is the outermost yellow half-circuit surrounding 0 in $A^+(r, R)$, and $\mathcal{C}_2(\omega)$ is the outermost yellow half-circuit surrounding 0 in the connected component of $A^+(r, R) \setminus \mathcal{C}_1(\omega)$ which is adjacent to $\Delta_i A^+(r, R)$, and so on. If n is odd we flip the colors of hexagons in $\mathcal{C}_2(\omega), \mathcal{C}_4(\omega), \dots, \mathcal{C}_{n-1}(\omega)$ and hexagons lie between $\Delta_i A^+(r, R)$ and $\mathcal{C}_n(\omega)$; if n is even we flip the colors of hexagons in $\mathcal{C}_2(\omega), \mathcal{C}_4(\omega), \dots, \mathcal{C}_n(\omega)$. It is easy to see that the resulting new configuration from the described color switching is in $\{N^+(r, R) = n\}$. But unfortunately, the map coming from such a color switching is not one-to-one and thus not a bijection. It turns out that one needs to flip the colors of more hexagons to construct a bijection. Namely, when n is odd, besides switching the colors of hexagons in $\mathcal{C}_2(\omega), \mathcal{C}_4(\omega), \dots, \mathcal{C}_{n-1}(\omega)$ and hexagons lie strictly between $\Delta_i A^+(r, R)$ and $\mathcal{C}_n(\omega)$, we also switch the colors of hexagons lie strictly between $\mathcal{C}_1(\omega)$ and $\mathcal{C}_2(\omega), \mathcal{C}_3(\omega)$ and $\mathcal{C}_4(\omega), \dots, \mathcal{C}_{n-2}(\omega)$ and $\mathcal{C}_{n-1}(\omega)$; When n is even, besides switching the colors of hexagons in $\mathcal{C}_2(\omega), \mathcal{C}_4(\omega), \dots, \mathcal{C}_n(\omega)$, we also switch the colors of hexagons lie between $\mathcal{C}_1(\omega)$ and $\mathcal{C}_2(\omega), \mathcal{C}_3(\omega)$ and $\mathcal{C}_4(\omega), \dots, \mathcal{C}_{n-1}(\omega)$ and $\mathcal{C}_n(\omega)$. One can check the map coming from this new color switching is a bijection. See Figure 2 for an example. We refer the reader to [22] for more details. \square

The following lemma is a large deviation bound for $T^+(r, R)$.

Lemma 1. *There exist constants $C_1, C_2 > 0$ and $K > 1$ such that for all $1 \leq r < R$ and $x > K \log_2(R/r)$,*

$$P(T^+(r, R) \geq x) \leq C_1 e^{-C_2 x}.$$

Proof. The proof is similar to that of Corollary 2.3 in [21]. \square

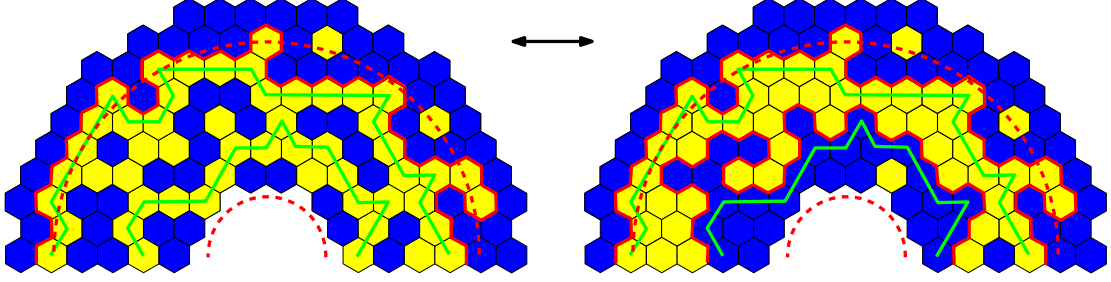


FIGURE 2. Color switching. The left annulus has two disjoint yellow half-circuits surrounding 0 while the right one has two disjoint interface half-loops surrounding 0.

2.2. Chordal SLE. We give a brief introduction of chordal Schramm-Loewner evolution (SLE). Please refer to [13] for more about SLE. Let $(B_t)_{t \geq 0}$ be a standard Brownian motion on \mathbb{R} with $B_0 = 0$. Let $\kappa \geq 0$ and consider the solution to the chordal Loewner equation for the upper half plane,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z, z \in \overline{\mathbb{H}}. \quad (1)$$

This is well defined as long as $g_t(z) - \sqrt{\kappa} B_t \neq 0$, i.e., for all $t < T_z$, where $T_z := \inf\{t \geq 0 : g_t(z) - \sqrt{\kappa} B_t = 0\}$. For each $t > 0$, $g_t : \mathbb{H} \setminus K_t \rightarrow \mathbb{H}$ is a conformal map, where $K_t := \{z \in \overline{\mathbb{H}} : T_z \leq t\}$ is a compact subset of $\overline{\mathbb{H}}$ such that $\mathbb{H} \setminus K_t$ is simply connected. It is known (see [16]) that $\gamma(t) := g_t^{-1}(\sqrt{\kappa} B_t)$ exists and continuous in t , and the curve γ is called the **trace** of chordal SLE_κ . It is also proven in the same paper that γ is simple if and only if $\kappa \in [0, 4]$.

Let D be a simply connected domain and a, b be distinct points on ∂D . Let $f : \mathbb{H} \rightarrow D$ be a conformal map with $f(0) = a$ and $f(\infty) = b$. If γ is the chordal SLE_κ trace in $\overline{\mathbb{H}}$ from 0 to ∞ , then $f \circ \gamma$ defines the chordal SLE_κ trace from a to b in \overline{D} .

3. CONFORMAL RADII

In [4], the scaling limit of the interface loops of critical site percolation on \mathcal{T} was constructed and proved. This scaling limit is the conformal loop ensembles (CLE) defined in [18] for $\kappa = 6$. In [17], the distribution of the conformal radii of the nested loops from CLE_κ ($8/3 \leq \kappa \leq 8$) in \mathbb{D} surrounding 0 was calculated. In this section, we are interested in the nested interface half-loops (in the scaling limit) surrounding 0 in \mathbb{D}^+ . We will compute the distribution of the “conformal radii” of those nested half-loops. Our derivation applies to any SLE_κ where $\kappa > 4$. But we can only obtain the explicit moment generating function for $\kappa = 6$, which is what we need for this paper. For $\kappa > 4$ and $\kappa \neq 6$, we show that it has the same distribution as the first hitting time of some SDE (see Remark 4 below), which might be interesting in itself.

Let $D_0 := \mathbb{D}^+$ be the upper half unit disk. Let $l_0 := \inf\{x : x \in \partial D_0 \cap \mathbb{R}\} = -1$ and $r_0 := \sup\{x : x \in \partial D_0 \cap \mathbb{R}\} = 1$. Suppose $\{\gamma(t), t \geq 0\}$ is the chordal SLE_6 trace in \overline{D}_0 from -1 to 0 and τ_0 is the first time t that 0 and r_0 are in distinct components of $\overline{D}_0 \setminus \gamma[0, t]$. Let D_1 be the connected component of $D_0 \setminus \gamma[0, \tau_0]$ that contains 0 as a boundary point. We inductively define (D_k, l_k, r_k) in the following way. For $k \in \mathbb{N}$, let $l_k := \inf\{x : x \in \partial D_k \cap \mathbb{R}\}$ and $r_k := \sup\{x : x \in \partial D_k \cap \mathbb{R}\}$. If k is odd, then denote by τ_k the first time t that 0 and l_k are in distinct components of $\overline{D}_k \setminus \gamma[\tau_{k-1}, t]$; If k is even, then denote by τ_k the first time t that 0 and r_k are in distinct components of

$\overline{D}_k \setminus \gamma[\tau_{k-1}, t]$. Let D_{k+1} be the connected component of $D_k \setminus \gamma[\tau_{k-1}, \tau_k]$ that contains 0 as a boundary point. If D is a simply connected domain and $z \in D$, define $\text{CR}(D, z)$ to be the **conformal radius** of D viewed from z , i.e., $\text{CR}(D, z) = |g'(z)|^{-1}$ where g is any conformal map from D to the unit disk \mathbb{D} that sends z to 0. For $k \in \mathbb{N} \cup \{0\}$, let \tilde{D}_k be the reflected domain of D_k , that is,

$$\tilde{D}_k = \{z \in \mathbb{C} : z \in D_k \text{ or } \bar{z} \in D_k\} \cup (l_k, r_k).$$

Define

$$Z_k := \log \text{CR}(\tilde{D}_{k-1}, 0) - \log \text{CR}(\tilde{D}_k, 0), \quad k \in \mathbb{N}.$$

We denote by \mathbb{P} the probability measure associated with the chordal SLE_6 in D_0 , and \mathbb{E} for the corresponding expectation. Then we have

Theorem 2. *The Z_k 's are i.i.d. random variables and*

$$\mathbb{E}e^{\lambda Z_k} = \frac{\sqrt{3}}{2 \cos(\pi \sqrt{1/36 + 2\lambda/3})},$$

where $\text{Re}(\lambda) < 1/3$.

Theorem 2 implies the following corollary immediately.

Corollary 2. *For $k \in \mathbb{N}$,*

$$\mathbb{E}Z_k = \frac{2\sqrt{3}\pi}{3}, \quad \text{Var}(Z_k) = \frac{16\pi^2}{3} - 8\sqrt{3}\pi.$$

The proof of Theorem 2 relies on ideas from [14] and [17]. We first introduce the following easy but useful lemma.

Lemma 2. *If $\phi : D \rightarrow D'$ is a conformal bijection and $z \in D$, then*

$$\text{CR}(D', \phi(z)) = |\phi'(z)| \text{CR}(D, z).$$

Proof. The proof follows from the definition of conformal radius. \square

Proof of Theorem 2. For k odd, by the domain Markov property and the locality property for SLE_6 (see, e.g., Proposition 6.14 in [13]), $\gamma[\tau_{k-1}, \tau_k]$ has the same distribution as a chordal SLE_6 trace in D_k from r_k to l_k stopped when it disconnects 0 from l_k . Similarly, for k even, $\gamma[\tau_{k-1}, \tau_k]$ has the same distribution as a chordal SLE_6 trace in D_k from l_k to r_k stopped when it disconnects 0 from r_k . The Z_k 's are i.i.d. follows from Lemma 2 and the conformal invariance of chordal SLE_6 (by definition). So it suffices to prove that Z_1 has the right moment generating function. Note that $f(z) = (\frac{z+1}{1-z})^2$ is a conformal bijection from D_0 to \mathbb{H} with $f(-1) = 0, f(0) = 1, f(1) = \infty$. So the proof of Theorem 2 is completed if we can show the following proposition, since $\text{CR}(\tilde{D}_1, 0) = \text{CR}(\tilde{U}_{T_1}, 1)/4$ where the latter is defined below. \square

Proposition 4. *Suppose $\{\gamma(t), t \geq 0\}$ is a chordal SLE_6 trace in $\overline{\mathbb{H}}$ from 0 to ∞ . For $z \in \overline{\mathbb{H}}$, let $T_z := \inf\{t \geq 0 : g_t(z) - \sqrt{6}B_t = 0\}$. Let $F_t := (-\infty, 0] \cup \{z \in \overline{\mathbb{H}} : T_z \leq t\}$ and $\tilde{F}_t = \{z : z \in F_t \text{ or } \bar{z} \in F_t\}$. For $t < T_1$, define \tilde{U}_t to be the connected component of $\mathbb{C} \setminus \tilde{F}_t$ that contains 1. Define $\text{CR}(\tilde{U}_{T_1}, 1) := \lim_{t \uparrow T_1} \text{CR}(\tilde{U}_t, 1)$ (see the Remark 3 below). Then we have*

$$\mathbb{E}e^{-\lambda \log(\text{CR}(\tilde{U}_{T_1}, 1)/4)} = \frac{\sqrt{3}}{2 \cos(\pi \sqrt{1/36 + 2\lambda/3})},$$

where $\text{Re}(\lambda) < 1/3$.

Remark 3. For $t < T_1$, define the inradius of \tilde{U}_t with respect to 1 as $\text{inrad}(\tilde{U}_t, 1) := \inf\{|z - 1| : z \notin \tilde{U}_t\}$. Then the Schwarz Lemma and the Koebe 1/4 Theorem give

$$\text{inrad}(\tilde{U}_t, 1) \leq \text{CR}(\tilde{U}_t, 1) \leq 4\text{inrad}(\tilde{U}_t, 1).$$

The equation (2) below implies $\text{CR}(\tilde{U}_t, 1)$ is strictly increasing for $t < T_1$, so the limit $\lim_{t \uparrow T_1} \text{CR}(\tilde{U}_t, 1)$ is well-defined and finite a.s.

Remark 4. For a general chordal SLE_κ where $\kappa > 4$, all quantities defined in the proposition and its proof are still well-defined. In this general case, (5) below becomes

$$d\theta_t = \frac{4}{5 \sin(\theta_t/2)} \left[3 - \frac{\kappa}{2} + \left(\frac{\kappa}{4} - 1 \right) \cos(\theta_t/2) \right] dt - \sqrt{\frac{4\kappa}{5}} dW_t, \text{ if } 0 < \theta_t < 2\pi.$$

Computing the exact distribution of the first hitting time of 2π for this process started at 0 might be hard.

Proof. As in the proof of Theorem 1 of [14], we define x_t to be the rightmost point of $\tilde{F}_t \cap \mathbb{R}$ and $g_t(x_t)$ is defined by

$$g_t(x_t) := \inf\{g_t(x) : x > 0, T_x > t\}.$$

Let $X_t = g_t(1) - \sqrt{6}B_t$, $O_t = g_t(x_t) - \sqrt{6}B_t$, $Y_t = X_t - O_t$, $J_t = Y_t/X_t$, $\Upsilon_t = \text{CR}(\tilde{U}_t, 1)/4$. For $t < T_1$, it is not hard to find a conformal bijection between \tilde{U}_t and \mathbb{D} . Using such a conformal bijection one gets

$$\Upsilon_t = \frac{Y_t}{g'_t(1)}.$$

By the Loewner equation (1), we have for $t < T_1$

$$\partial_t \Upsilon_t = \frac{-2\Upsilon_t J_t}{X_t^2(1 - J_t)}, \quad (2)$$

$$dJ_t = \frac{J_t}{X_t^2} \left(4 - \frac{2}{1 - J_t} \right) dt + \frac{\sqrt{6}J_t}{X_t} dB_t. \quad (3)$$

Define the random time change

$$\sigma(t) = \inf\{s : s \geq 0, \Upsilon_s = e^{-2t/5}\}. \quad (4)$$

We also define $\hat{\Upsilon}_t = \Upsilon_{\sigma(t)} = e^{-2t/5}$, $\hat{X}_t = X_{\sigma(t)}$, $\hat{J}_t = J_{\sigma(t)}$. The equation (2) and the chain rule imply

$$-\frac{2}{5}\hat{\Upsilon}_t = -\frac{2}{5}e^{-2t/5} = \partial_t \hat{\Upsilon}_t = \dot{\sigma}(t) \frac{-2\hat{\Upsilon}_t \hat{J}_t}{\hat{X}_t^2(1 - \hat{J}_t)}.$$

Therefore,

$$\dot{\sigma}(t) = \frac{\hat{X}_t^2(1 - \hat{J}_t)}{5\hat{J}_t}.$$

We change time in (3) to get

$$d\hat{J}_t = \frac{2 - 4\hat{J}_t}{5} dt + \sqrt{\frac{6\hat{J}_t(1 - \hat{J}_t)}{5}} dW_t,$$

where $W_t = \int_0^{\sigma(t)} \frac{1}{\sqrt{\dot{\sigma}(\sigma^{-1}(s))}} dB_s$ is a standard Brownian motion. If we make the change of variables $\hat{J}_t = \frac{1 + \cos(\theta_t/2)}{2}$, then Itô's formula implies

$$d\theta_t = \frac{2}{5} \cot(\theta_t/2) dt - \sqrt{\frac{24}{5}} dW_t, \quad \theta_0 = 0, \text{ if } 0 < \theta_t < 2\pi. \quad (5)$$

Note that θ_t behaves like a Bessel process, and it is reflected instantaneously at 0 (reflected in the same way that the Bessel process is reflected). It is easy to see that $J_0 = \hat{J}_0 = 1$ and $\lim_{t \uparrow T_1} J_t = 0$. Let $\tau_0 := \inf\{s : \hat{J}_s = 0\}$ and $S_{2\pi} := \inf\{s : \theta_s = 2\pi\}$. Then the definition of θ_t and (4) give

$$\tau_0 \stackrel{d}{=} S_{2\pi}, \quad \Upsilon_{T_1} \stackrel{d}{=} e^{-2\tau_0/5}. \quad (6)$$

Our θ_t defined in (5) has the same distribution as θ_t for $\kappa = 24/5$ defined in equation (6) of [17]. Hence Proposition 2 and the equation (3) from [17] say

$$\mathbb{E}e^{\lambda S_{2\pi}} = \frac{\sqrt{3}}{2 \cos(\pi \sqrt{1/36 + 5\lambda/3})}, \quad \operatorname{Re}(\lambda) < \frac{2}{15}.$$

The above displayed equation and (6) imply

$$\mathbb{E}e^{-\lambda \log(\operatorname{CR}(\tilde{U}_{T_1,1})/4)} = \mathbb{E}e^{-\lambda \log(\Upsilon_{T_1})} = \mathbb{E}e^{(2\lambda/5)S_{2\pi}} = \frac{\sqrt{3}}{2 \cos(\pi \sqrt{1/36 + 2\lambda/3})}, \quad \operatorname{Re}(\lambda) < \frac{1}{3},$$

which completes the proof of the proposition. \square

4. FIRST PROOF OF SLLW USING CONFORMAL RADII

Our first proof of the strong law of large numbers for c_n^+ using the conformal radii result that we proved in the last section. Recall the definition of \tilde{D}_k in the previous section. For $\epsilon \in (0, 1)$, we define

$$N(\epsilon) := \sup\{k : \overline{\mathbb{D}}_\epsilon \subseteq \tilde{D}_k\}.$$

Theorem 2 and its corollary enable us to show the following

Proposition 5.

$$\lim_{\epsilon \downarrow 0} \frac{N(\epsilon)}{-\log(\epsilon)} = \frac{\sqrt{3}}{2\pi} \text{ a.s.}, \quad \lim_{\epsilon \downarrow 0} \frac{\mathbb{E}N(\epsilon)}{-\log(\epsilon)} = \frac{\sqrt{3}}{2\pi}, \quad \lim_{\epsilon \downarrow 0} \frac{\operatorname{Var}(N(\epsilon))}{-\log(\epsilon)} = \frac{2\sqrt{3}}{\pi} - \frac{9}{\pi^2}.$$

Proof. The proof uses some basic properties for renewal processes and a similar proof can be found in Proposition 3.2 of [22]. \square

Next, we prove the scaling limit of $T^+(\rho r, \rho R)$ as $\rho \rightarrow \infty$.

Proposition 6. *Suppose $1 \leq r < R$, $\rho \geq 1$ and $k \in \mathbb{N}$. Assume that hexagons in $\Delta_o A^+(\rho r, \rho R)$ are colored blue. We have*

$$T^+(\rho r, \rho R) \rightarrow N(r/R) \text{ in distribution as } \rho \rightarrow \infty, \\ E(T^+(\rho r, \rho R)^k) \rightarrow \mathbb{E}(N(r/R)^k) \text{ as } \rho \rightarrow \infty.$$

Proof. Recall \mathbb{D}^+ is the upper half unit disk. Denote by \mathbb{D}_δ^+ the smallest connected domain of hexagons (in \mathcal{H}_δ) containing $\overline{\mathbb{D}^+}$. Let $\partial\mathbb{D}_\delta^+$ be the topological boundary of \mathbb{D}_δ^+ (here \mathbb{D}_δ^+ is considered as a domain of \mathbb{C}) and $\Delta\mathbb{D}_\delta^+$ be the external site boundary of \mathbb{D}_δ^+ (i.e., the set of hexagons that do not belong to \mathbb{D}_δ^+ but are adjacent to hexagons in \mathbb{D}_δ^+). A vertex $x \in \partial\mathbb{D}_\delta^+$ is called an **e-vertex** if the edge containing x that is not in $\partial\mathbb{D}_\delta^+$ does not belong to \mathbb{D}_δ^+ neither. Let $(-1)_\delta$ (0_δ , respectively) be a closest e-vertex of \mathcal{H}_δ to -1 (0 , respectively). Denote by $\partial_{-1,0}\mathbb{D}_\delta^+$ the portion of $\partial\mathbb{D}_\delta^+$ traversed counterclockwise from $(-1)_\delta$ to 0_δ , and the portion of $\Delta\mathbb{D}_\delta^+$ whose hexagons are adjacent to $\partial_{-1,0}\mathbb{D}_\delta^+$ is denoted by $\Delta_{-1,0}\mathbb{D}_\delta^+$. The remaining part of $\Delta\mathbb{D}_\delta^+$ is denoted by $\Delta_{0,-1}\mathbb{D}_\delta^+$. Suppose we color yellow all hexagons in $\Delta_{-1,0}\mathbb{D}_\delta^+$ and blue all those in $\Delta_{0,-1}\mathbb{D}_\delta^+$. Then for any percolation configuration inside \mathbb{D}_δ^+ , there is a unique interface path (say γ_δ) from $(-1)_\delta$ to 0_δ , which separates the yellow cluster adjacent to $\Delta_{-1,0}\mathbb{D}_\delta^+$ from the blue cluster adjacent

to $\Delta_{0,-1}\mathbb{D}_\delta^+$. The random path γ_δ is called a **chordal exploration path** in D_δ from $(-1)_\delta$ to 0_δ . We remark that γ_δ does not depend on the color of hexagons in $\Delta\mathbb{D}_\delta^+$. It is well-known that γ_δ converges weakly to a chordal SLE₆ trace in \mathbb{D}_δ^+ from -1 to 0 (see [19] and [5]).

By using 3-arm event in the half-plane, it is not hard to show that whenever γ_δ comes close to the boundary $\partial\mathbb{D}_\delta^+$ then it does touch the boundary with high probability (see, e.g., Lemma 6.1 of [4]). This implies that the number of interface half-loops surrounding $(r/R)\mathbb{D}_\delta^+$ in \mathbb{D}_δ^+ converges weakly to $N(r/R)$ as $\delta \rightarrow 0$. Therefore $N^+(\rho r, \rho R)$ converges weakly to $N(r/R)$ as $\rho \rightarrow \infty$.

Now Proposition 3 implies $T^+(\rho r, \rho R)$ converges weakly to $N(r/R)$ as $\rho \rightarrow \infty$. Lemma 1 says that $\{T^+(\rho r, \rho R)\}_{\rho \geq 1}$ is uniformly integrable, so

$$E(T^+(\rho r, \rho R)^k) \rightarrow \mathbb{E}(N(r/R)^k) \text{ as } \rho \rightarrow \infty.$$

□

We are ready to prove the strong law of large numbers for c_n^+ .

Proposition 7.

$$\lim_{n \rightarrow \infty} \frac{c_n^+}{\log n} = \frac{\sqrt{3}}{2\pi} a.s., \quad \lim_{n \rightarrow \infty} \frac{Ec_n^+}{\log n} = \frac{\sqrt{3}}{2\pi}.$$

Proof. The proof uses Propositions 5 and 6, and is similar to that of Proposition 3.6 of [22]. □

5. SECOND PROOF OF SLLW USING EXPECTED NUMBER OF CLUSTERS

In [10], an explicit formula for the scaling limit of the expected number of clusters crossing a Jordan domain is given, and this scaling limit is proved to be conformal invariant. Using that result and Proposition 3, we give a second proof of Proposition 7.

Let $D \subsetneq \mathbb{C}$ be a Jordan domain (i.e., D is simply connected and the boundary of D , ∂D , is a Jordan curve). Let $z_1, z_2, z_3, z_4 \in \partial D$. We assume ∂D is oriented counterclockwise, and z_1, z_2, z_3, z_4 appear in this order. Suppose $\phi : D \rightarrow \mathbb{H}$ is a conformal map. The **cross-ratio** of $(D; z_1, z_2, z_3, z_4)$ is defined by

$$\lambda := \lambda(D; z_1, z_2, z_3, z_4) = \frac{(\phi(z_4) - \phi(z_3))(\phi(z_2) - \phi(z_1))}{(\phi(z_4) - \phi(z_2))(\phi(z_3) - \phi(z_1))}.$$

It is easy to see that Möbius transformations preserve cross-ratios, $\lambda \in (0, 1)$ and

$$\lambda(D; z_1, z_2, z_3, z_4) = 1 - \lambda(D; z_2, z_3, z_4, z_1).$$

We state a result by Hongler and Smirnov [10]. Recall D_δ is the smallest connected domain of hexagons (in \mathcal{H}_δ) which contains \overline{D} . Consider critical site percolation on D_δ . Recall that P_δ and E_δ are corresponding probability measure and expectation. Let z_i^δ be a closest vertex of ∂D_δ to z_i for $i = 1, 2, 3, 4$. Let $N(D_\delta; z_1^\delta, z_2^\delta, z_3^\delta, z_4^\delta)$ be the number of open clusters in D_δ which connect the arc along ∂D_δ from z_1^δ to z_2^δ and the arc along ∂D_δ from z_3^δ to z_4^δ . Then we have

Theorem 3 ([10]). *Let $D \subsetneq \mathbb{C}$ be a Jordan domain and $z_1, z_2, z_3, z_4 \in \partial D$ are ordered counterclockwise. Suppose λ is the cross-ratio of $(D; z_1, z_2, z_3, z_4)$. Then we have*

$$\begin{aligned} \lim_{\delta \rightarrow 0} E_\delta(N(D_\delta; z_1^\delta, z_2^\delta, z_3^\delta, z_4^\delta)) &= \frac{2\pi\sqrt{3}}{\Gamma(1/3)} \lambda^{\frac{1}{3}} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; \lambda\right) - \frac{\sqrt{3}}{4\pi} \lambda {}_3F_2\left(1, 1, \frac{4}{3}; \frac{5}{3}, 2; \lambda\right) \\ &\quad + \frac{\sqrt{3}}{4\pi} \log\left(\frac{1}{1-\lambda}\right). \end{aligned}$$

Remark 5. By the argument leads to the above theorem in [10], one sees that

$$\left| \frac{2\pi\sqrt{3}}{\Gamma(1/3)} \lambda^{\frac{1}{3}} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; \lambda\right) - \frac{\sqrt{3}}{4\pi} \lambda {}_3F_2\left(1, 1, \frac{4}{3}; \frac{5}{3}, 2; \lambda\right) \right| \leq 1.$$

Let $a^+(1, R) := \{z \in \mathbb{H} : 1 < |z| < R\}$ for some $R > 1$ be the half-annulus in the upper half-plane with inner radius 1 and outer radius R . Let $z_1 = -R, z_2 = -1, z_3 = 1, z_4 = R \in \partial a^+(1, R)$. Then we have

Proposition 8.

$$\lim_{R \rightarrow \infty} \frac{\lim_{\delta \rightarrow 0} E_\delta(N(a^+(1, R)_\delta; z_1^\delta, z_2^\delta, z_3^\delta, z_4^\delta))}{\log R} = \frac{\sqrt{3}}{4\pi}$$

Proof. Let $D((\log R)/\pi) := \{z \in \mathbb{C} : 0 < \operatorname{Re}(z) < (\log R)/\pi, 0 < \operatorname{Im}(z) < 1\}$. Then $\phi : a^+(1, R) \rightarrow D((\log R)/\pi)$ such that $\phi(z) = (\log z)/\pi$ is a conformal map. Hence

$$\begin{aligned} \lambda(a^+(1, R); z_1, z_2, z_3, z_4) &= \lambda(D((\log R)/\pi); (\log R)/\pi + i, i, 0, (\log R)/\pi) \\ &= 1 - \lambda(D((\log R)/\pi); i, 0, (\log R)/\pi, (\log R)/\pi + i). \end{aligned}$$

The proposition follows from Theorem 3, Remark 5 and the following lemma. \square

Lemma 3. Let η be the **aspect-ratio** of the rectangle $D(\eta) := \{z \in \mathbb{C} : 0 < \operatorname{Re}(z) < \eta, 0 < \operatorname{Im}(z) < 1\}$, i.e., the ratio of the width of the rectangle to its height. Then

$$\lim_{\eta \rightarrow \infty} \frac{\log \lambda(D(\eta))}{\eta} = -\pi,$$

where $\lambda(D(\eta))$ is the cross-ratio of $(D(\eta); i, 0, \eta, \eta + i)$.

Proof. To be consistent with the tradition in the literature, we will denote by k for some real number in $(0, 1)$ in the proof. Let $R(k)$ be the rectangle with corners $\pm K(k^2), \pm K(k^2) + iK(1 - k^2)$ where

$$K(u) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-ut^2)}}$$

is the complete elliptic integral of the first kind. Let $\psi : \mathbb{H} \rightarrow R(k)$ be the Schwartz-Christoffel transform

$$\psi(z) = \int_0^z \frac{dz}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Then ψ is a conformal map with $\psi(-k^{-1}) = -K(k^2) + iK(1 - k^2)$, $\psi(-1) = -K(k^2)$, $\psi(1) = K(k^2)$ and $\psi(k^{-1}) = K(k^2) + iK(1 - k^2)$. So

$$\lambda(R(k); -K(k^2) + iK(1 - k^2), -K(k^2), K(k^2), K(k^2) + iK(1 - k^2)) = \frac{(1 - k)^2}{(1 + k)^2}.$$

Note that the aspect-ratio for $R(k)$ is $2K(k^2)/K(1 - k^2)$. To prove the lemma, it suffices to show

$$\lim_{k \uparrow 1} \frac{\log((1 - k)^2/(1 + k)^2)}{2K(k^2)/K(1 - k^2)} = -\pi.$$

The last limit is true because of the well-known estimates (see, e.g. p. 57 of [15])

$$K(1 - k^2) = \frac{\pi}{2} + o(1), \quad K(k^2) = 2 \log 2 - \log \sqrt{1 - k^2} + o(1), \quad \text{as } k \uparrow 1.$$

\square

Now we can give a second proof of the strong law of large numbers for c_n^+ (i.e., Proposition 7). For any $k, j \in \mathbb{N}$, let us put monochromatic (blue) boundary condition on $A^+(2^{k(j-1)}, 2^{kj})$ (i.e., color blue all hexagons in $\Delta A^+(2^{k(j-1)}, 2^{kj})$). In the rest of the section, we will fix the lattice spacing to be 1 (i.e., $\delta = 1$) and enlarge the domain to get a scaling limit. It is easy to see from Proposition 3 that

$$T^+(2^{k(j-1)}, 2^{kj}) \stackrel{d}{=} N^+(2^{k(j-1)}, 2^{kj}) = 2N(A^+(2^{k(j-1)}, 2^{kj}), -2^{kj}, -2^{k(j-1)}, 2^{k(j-1)}, 2^{kj}).$$

Then Theorem 3 and Proposition 8 imply that

- $\lim_{j \rightarrow \infty} ET^+(2^{k(j-1)}, 2^{kj})$ exists and the limit only depends on $1/2^k$.
-

$$\lim_{k \rightarrow \infty} \frac{\lim_{j \rightarrow \infty} ET^+(2^{k(j-1)}, 2^{kj})}{\log 2^k} = \frac{\sqrt{3}}{2\pi}.$$

If one checks the proof of Proposition 3.6 in [22] carefully, one sees that the above two items are enough to prove Proposition 7.

6. VARIANCE OF c_n^+

The proof of the limit result for $\text{Var}(c_n^+)$ is essentially the same as for $\text{Var}(c_n)$ in [22]. For the convenience of the reader, we give the idea in the following. As in [22], we use a modified martingale method introduced in [12]. We first introduce some notations. For $j \in \mathbb{N} \cup \{0\}$, define the half-annulus

$$A^+(j) := A^+(2^j, 2^{j+1}).$$

Furthermore, define

$$\begin{aligned} m(j) &:= \inf\{k \geq j : A(k) \text{ contains a blue half-circuit surrounding } 0\}, \\ \mathcal{C}_j &:= \text{the innermost blue half-circuit surrounding } 0 \text{ in } A^+(m(j)), \\ \mathcal{F}_j &:= \sigma\text{-field generated by } \{t(v) : v \in \overline{\mathcal{C}}_j\}, \end{aligned}$$

where $\overline{\mathcal{C}}_j := \{\text{the sites in the finite component of } \{\mathbf{V} \cap \overline{\mathbb{H}}\} \setminus \mathcal{C}_j\} \cup \mathcal{C}_j$. Denote by $T^+(0, \mathcal{C}_j)$ the first-passage time in $\overline{\mathcal{C}}_j$ from 0 to \mathcal{C}_j . That is,

$$T^+(0, \mathcal{C}_j) := \inf\{T(\gamma) : \gamma \in \overline{\mathcal{C}}_j \text{ and } \gamma \text{ starts at } 0 \text{ and ends at a site in } \mathcal{C}_j\}.$$

For all $j \in \mathbb{N}$, denote by \mathcal{F}_{-j} the trivial σ -field. For $k, q \in \mathbb{N}$, write

$$T^+(0, \mathcal{C}_{kq}) - E(T^+(0, \mathcal{C}_{kq})) = \sum_{j=0}^q (E(T^+(0, \mathcal{C}_{kq}) | \mathcal{F}_{kj}) - E(T^+(0, \mathcal{C}_{kq}) | \mathcal{F}_{k(j-1)})) := \sum_{j=0}^q \Delta_{k,j}.$$

Then $\{\Delta_{k,j}\}_{0 \leq j \leq q}$ is a \mathcal{F}_{kj} -martingale increment sequence. Hence,

$$\text{Var}(T^+(0, \mathcal{C}_{kq})) = \sum_{j=0}^q E(\Delta_{k,j}^2).$$

One can use this sum to estimate $\text{Var}[c_n^+]$ with $2^{kq} \leq n \leq 2^{k(q+1)}$. The proof proceeds as follows. First, essentially in the same way one can prove the half-circuit version of Lemmas 3.7, 3.8, 3.9, 3.10 in [22]. Then, similarly to the proof of Proposition 3.11 in [22], combining these Lemmas, Propositions 5 and 6, one gets the following proposition.

Proposition 9.

$$\lim_{n \rightarrow \infty} \frac{\text{Var}(c_n)}{\log n} = \frac{2\sqrt{3}}{\pi} - \frac{9}{\pi^2}.$$

7. PROOFS OF THEOREM 1, COROLLARY 1, PROPOSITIONS 1 AND 2

In this section, we complete the proofs of all results stated in the introduction.

Proof of Theorem 1. Theorem 1 is just the combination of Propositions 7 and 9. \square

Proof of Corollary 1. Recall the definitions of \tilde{D}_k and $N(\epsilon)$ in Sections 3 and 4. We can define \tilde{D}_k^α similarly by running a chordal SLE $_\alpha$ in $\overline{\mathbb{D}^\alpha}$ from $e^{i\alpha}$ to 0. Let

$$N^\alpha(\epsilon) := \sup\{k : \overline{\mathbb{D}_\epsilon} \subseteq \tilde{D}_k^\alpha\}.$$

Note that $\phi(z) = z^{\pi/\alpha}$ is a conformal bijection from \mathbb{D}^α to \mathbb{D}^+ . Using the conformal invariance of chordal SLE, we have

$$N^\alpha(\epsilon) \stackrel{d}{=} N(\epsilon^{\pi/\alpha}).$$

So Proposition 5 implies

$$\lim_{\epsilon \downarrow 0} \frac{N^\alpha(\epsilon)}{\log(1/\epsilon)} = \frac{\sqrt{3}}{2\alpha} a.s., \quad \lim_{\epsilon \downarrow 0} \frac{\mathbb{E}N^\alpha(\epsilon)}{\log(1/\epsilon)} = \frac{\sqrt{3}}{2\alpha}, \quad \lim_{\epsilon \downarrow 0} \frac{\text{Var}(N^\alpha(\epsilon))}{\log(1/\epsilon)} = \frac{2\sqrt{3}}{\alpha} - \frac{9}{\pi\alpha}.$$

The rest of the proof is similar to that of Theorem 1. \square

Proof of Proposition 1. Note that the limit (in probability) of $T_{D_\delta}(a_\delta, b_\delta)$ as $\delta \downarrow 0$ only depends on the local geometry near a and b . So one can approximate the portion of ∂D near a by sequences of secant lines and/or tangent lines. More precisely, suppose $\beta(t_0) = a$ where $t_0 > 0$. Then one can pick $\{\epsilon_k^u, k \geq 1\}$, $\{\epsilon_k^d, k \geq 1\}$, $\{\eta_k^u, k \geq 1\}$, $\{\eta_k^d, k \geq 1\}$ where $\epsilon, \eta \geq 0$ and $\epsilon_k \downarrow 0, \eta_k \downarrow 0$ as $k \rightarrow \infty$ such that: if we denote by l_k^u (resp., l_k^d) the secant line segment between $\beta(t_0)$ and $\beta(t_0 - \eta_k^u)$ (resp., $\beta(t_0 - \eta_k^d)$) (if $\eta_k = 0$ we set l_k to be the left tangent line at a), and by r_k^u (resp., r_k^d) the secant line segment between $\beta(t_0)$ and $\beta(t_0 + \epsilon_k^u)$ (resp., $\beta(t_0 + \epsilon_k^d)$) (if $\epsilon_k = 0$ we set r_k to be the right tangent line at a), then when we replace the portion of ∂D at $\beta[t_0 - \eta_k^u, t_0 + \epsilon_k^u]$ by $l_k^u \cup r_k^u$ the resulting new domain is smaller (could be equal) and when we replace the portion of ∂D at $\beta[t_0 - \eta_k^d, t_0 + \epsilon_k^d]$ by $l_k^d \cup r_k^d$ the resulting new domain is larger (could be equal). The angle subtended by l_k^u and r_k^u (resp., l_k^d and r_k^d) is denoted by Θ_k^u (resp., Θ_k^d). Let $\Gamma \subset D$ be a simple curve such that $a, b \notin \Gamma$, a and b are boundary points in different connected components of $D \setminus \Gamma$. Let Γ_δ be the lattice approximation of Γ . Define $T_{D_\delta}(a_\delta, \Gamma_\delta)$ to be the first passage time in D_δ between a_δ and Γ_δ . Define $T_{D_\delta}^u(a_\delta, \Gamma_\delta)_k$ (resp., $T_{D_\delta}^d(a_\delta, \Gamma_\delta)_k$) to be the first passage time in the discrete approximation of the domain with boundary $(\partial D \setminus \beta[t_0 - \eta_k^u, t_0 + \epsilon_k^u]) \cup l_k^u \cup r_k^u$ (resp., $(\partial D \setminus \beta[t_0 - \eta_k^d, t_0 + \epsilon_k^d]) \cup l_k^d \cup r_k^d$) between a_δ and Γ_δ . Then clearly

$$T_{D_\delta}^d(a_\delta, \Gamma_\delta)_k \leq T_{D_\delta}(a_\delta, \Gamma_\delta) \leq T_{D_\delta}^u(a_\delta, \Gamma_\delta)_k.$$

Corollary 1 implies that

$$\begin{aligned} \lim_{\delta \downarrow 0} \frac{T_{D_\delta}^u(a_\delta, \Gamma_\delta)_k}{-\log \delta} &= \frac{\sqrt{3}}{2\Theta_k^u} a.s., & \lim_{\delta \downarrow 0} \frac{T_{D_\delta}^d(a_\delta, \Gamma_\delta)_k}{-\log \delta} &= \frac{\sqrt{3}}{2\Theta_k^d} a.s., \\ \lim_{\delta \downarrow 0} \frac{E_\delta T_{D_\delta}^u(a_\delta, \Gamma_\delta)_k}{-\log \delta} &= \frac{\sqrt{3}}{2\Theta_k^u}, & \lim_{\delta \downarrow 0} \frac{E_\delta T_{D_\delta}^d(a_\delta, \Gamma_\delta)_k}{-\log \delta} &= \frac{\sqrt{3}}{2\Theta_k^d}. \end{aligned}$$

It is clear that $\Theta_k^u \rightarrow \Theta_D(a)$ and $\Theta_k^d \rightarrow \Theta_D(a)$ as $k \rightarrow \infty$. Hence

$$\lim_{\delta \downarrow 0} \frac{T_{D_\delta}(a_\delta, \Gamma_\delta)}{-\log \delta} = \frac{\sqrt{3}}{2\Theta_D(a)} a.s., \quad \lim_{\delta \downarrow 0} \frac{E_\delta T_{D_\delta}(a_\delta, \Gamma_\delta)}{-\log \delta} = \frac{\sqrt{3}}{2\Theta_D(a)}.$$

Similar limits hold for $T_{D_\delta}(b_\delta, \Gamma_\delta)$. A standard argument as in (2.84) of [12] should yield

$$\lim_{\delta \downarrow 0} \frac{T_{D_\delta}(a_\delta, b_\delta) - T_{D_\delta}(a_\delta, \Gamma_\delta) - T_{D_\delta}(b_\delta, \Gamma_\delta)}{-\log \delta} = 0 \text{ in probability.}$$

It is not hard to show the fractional expression of the last equation viewed as a sequence of δ is uniformly integrable. This completes the proof. \square

Proof of Proposition 2. The basic idea is to show that both $E|c_n^+ - s_{0,n}|$ and $E|c_n^+ - s_{0,n}|^2$ are bounded up by some constant. Since the proof is very similar to the proof of Theorem 1.1 in [22], we omit the details here. \square

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